Diagonal Arguments in CCC

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Contents

Notes on DIAGONAL ARGUMENTS AND CARTESIAN CLOSED CATEGORIES by F. WILLIAM LAWVERE.

1 Background

Definition: A Cartesian closed category (CCC) is a category C with the following three adjoints:

- A right adjoint to $\mathcal{U}$ the unique functor into $\mathbf{1}$, the category with one object and one morphism;

$$\mathcal{U} : C \leftrightarrows \mathbf{1} : 1$$

- A right adjoint to the diagonal functor

$$\Delta : C \leftrightarrows C \times C : \times$$

- For each $A \in C$ a right adjoint

$$A \times (-) : C \leftrightarrows C : (-)^A$$

Recall that for an adjunction, $L \dashv R : C \to \mathcal{D}$ the unit and counit are natural transformations $\eta : id_C \to RL$ (unit) $\epsilon : LR \to id_{\mathcal{D}}$ (counit), making some diagrams commute.

For products we have the unit $\delta : id_C \to \times \circ \Delta$ and counit $\pi : \Delta \circ \times \to id_{C \times C}$. In particular the unit is important for us and can be described explicitly as

$$\delta_X : X \to X \times X; x \mapsto (x, x)$$

For each exponential adjunction, $A \times (-) \dashv (-)^A$, there is a unit $\lambda_A : id_C \to (A \times -)^A$ and co-unit $\epsilon_A : A \times (-)^A \to id_C$. In the category of sets the unit is "pairing with an element"

$$\begin{aligned} \lambda_A(X) : & X \to (A \times X)^A \\ & x \mapsto (\varphi : A \to A \times X) \\ & \qquad a \mapsto (a, x) \end{aligned}$$

And the co-unit is evaluation

$$\begin{aligned} \epsilon_A(X) : & A \times X^A \to X \\ & (a, \varphi) \mapsto \varphi(a) \end{aligned}$$

Because in the cateory of sets the internal hom is sincerely the Hom.

Definition: For a morphism $f : A \times X \to Y$ we define its λ-transform (or "Currying") as the composition $f^A \circ (\lambda_A(X))$

$$X \xrightarrow{\lambda_A(X)} (A \times X)^A \xrightarrow{f^A} Y^A$$

If we have an $f : A \times X \to Y$ and denote its λ-transform as $\bar{f} : A \to Y^A$ then if we are in the category of sets

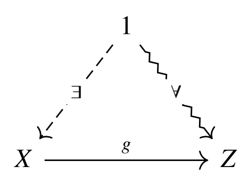
$$f(a, x) = \bar{f}(x)(a)$$

The first adjunction in the definition of a CCC guarantees the existence of some (unique up to isomorphism) terminal object that we call (overloading the notation with the adjunction itself) simply 1.

Relation to generators of a category and the fact that in sets $Hom(1, X) \cong X$

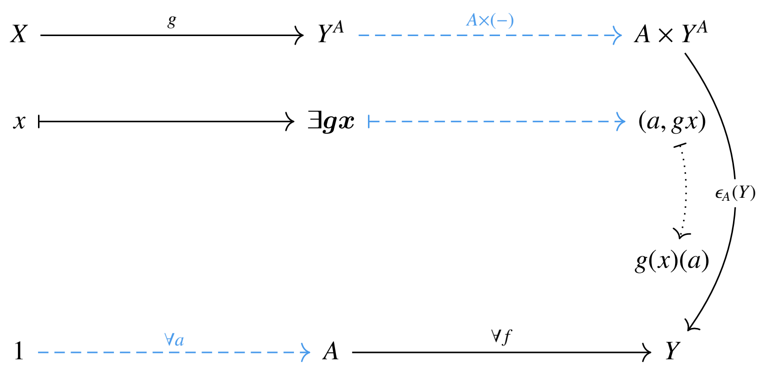
Definition: $g : X \to Z$ is point surjective iff for every $z : 1 \to Z$ there is some $x : 1 \to X$ such that $g \circ x = z$ i.e.

$$g : X \to Z \text{ point surjective} \iff (\forall z \in Hom_C(1, Z))(\exists x \in Hom_C(1, X))(g \circ x = z)$$



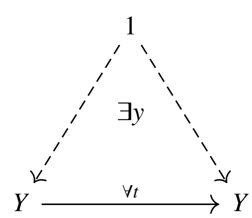
Definition: A map $g : X \to Y^A$ is weakly point surjective iff for every map $f : A \to Y$ there is some $x : 1 \to X$ such that on every $a : 1 \to A$ we have that $\epsilon(a, g \circ x) = f \circ a$ i.e.

$$g : X \to Y^A \text{ weakly point surjective} \iff (\forall f \in Hom_C(A, Y))(\exists x \in Hom_C(1, X))(\forall a \in Hom_C(1, A))(\epsilon(a, g\circ x) = f\circ a)$$



Definition: An object Y (in a category C) has the fixed point property iff for every endomorphism of Y there is some $y : 1 \to Y$ such that $t \circ y = y$ i.e.

$$Y \in C \text{ has fixed point property} \iff (\forall t \in Hom_C(Y, Y))(\exists y \in Hom_C(1, Y))(t \circ y = y)$$



2 Fixed Point Theorem

Lemma 1 *morphisms $A \times X \to Y$ are uniquely determined by their λ-transforms and visa versa.*

The claim can be rephrased as: For an $f : A \times X \to Y$ we have that $h : X \to Y^A$ is the λ-transform of f iff the following diagram commutes

$$\begin{array}{ccc} A \times X & & \\ {\scriptstyle A\times h}\downarrow & \searrow^{f} & \\ A \times Y^A & \xrightarrow[\epsilon_A(Y)]{} & Y \end{array}$$

Finish the proof....

The condition of weak surjectivity can be rephrased in light of the lemma. For any $g : A \to Y^A$ weakly surjective we can look at the map $\bar{g}$ whose λ-transform is g. Then for a, x as in the weak surjectivity definition, $\epsilon(a, g \circ x) = \bar{g}(a, x)$ by the diagram in 1. Then we see that weak surjectivity of g is equivilient to the following condition on $\bar{g}$: for any $f : A \to Y$ there is an $x : 1 \to A$ such that for all $a : 1 \to A$

$$\bar{g}(a, x) = f \circ a$$

i.e. g is weakly point surjective onto the internal hom, Y^A, if every function, $f \in Hom(A, Y)$, f is represented by $g(x)$ (up to what the terminal object sees and for some x).

Note that this is effectively the definition used in the proof and so we could rephrase this theorem to be in a category where only finite products exist.

Theorem 1 *In a Cartesian closed category C and for an arbitrary object Y, if there exists some object A and a weak point surjective morphism $g : A \to Y^A$ then Y has the fixed point property.*

Proof. Let A and $g : A \to Y^A$ be given as in the theorem. Now by the lemma there is a map $\bar{g} : A \times A \to Y$ whose lambda transform is g.

Now let $t \in Hom(Y, Y)$ arbitrary. If we can give a fixed point of t then we will be done. So let $f : A \to Y$ be the map

$$A \xrightarrow{\delta_A} A \times A \xrightarrow{\bar{g}} Y \xrightarrow{t} Y$$

$$a \mapsto (a, a) \mapsto \bar{g}(a, a) \mapsto t \circ \bar{g}(a, a)$$

Then by weak surjectivity of g there is an x such that for all a

$$\bar{g}(a, x) = f(a) = t \circ \bar{g}(a, a)$$

Then $\bar{g}(x, x)$ is a fixed point of t.

Perhaps the contrapositive of the theorem is more digestible: If there is a map $t : Y \to Y$ that has no fixed points then there can be no (weak point) surjections $A \to Y^A$ (for any A).

3 Examples

Theorem 2 *$\mathbb{R}$ is uncountable.*

Proof. Note that we are working in the category of $\underline{Set}$. It will suffice to show that there is no surjection $\mathbb{N} \to \mathbb{R}$.

Let $2 = \{0, 1\}$ then we notice that 2 does not have the fixed-point property because we have $t : 2 \to 2$ sending $1 \mapsto 0,\ 0 \mapsto 1$.

Thus by the (contrapositive of the) theorem there are no surjections $\mathbb{N} \to 2^{\mathbb{N}} \cong \mathbb{R}$

Theorem 3 *For any set S*

$$|S| < |\mathcal{P}(S)|$$

where $\mathcal{P}$ denotes the powerset

Proof. It suffices to prove that there is no surjection $S \to \mathcal{P}(S)$.

Again we know by the theorem that there is no surjection $S \to 2^S \cong \mathcal{P}(S)$ so we are done.

These proofs both rely on the fact that $\mathbb{R} \cong 2^{\mathbb{N}}$ or that $\mathcal{P}(S) \cong 2^S$. These facts are easy to prove however because we are in the category of sets where $2^S = Hom(S, 2)$. Thus the arguments go that there is a clear bijection between these things and functions out of 2.

For $\mathbb{R}$we the bijection to the Hom set is clear because any real number has a binary representation. For the powerset we think that for any subset there are two choices either inclusion or exclusion.

Theorem 4 (Russell's Paradox) *(The lambda transform of) Set theoretic membership is not a weak point surjection.*

Proof. Consider $\in: A \times A \to 2$ the set theoretic membership relation. Now assume that this is a weak point surjection (of its lambda transform), i.e. $\forall f : A \to 2$ we have that there is some $x \in A, \in (a, x) = f(a)$.

Then set $f = t \circ \in \circ \delta_A$ where t is the map $t : 2 \to 2$ sending $1 \mapsto 0,\ 0 \mapsto 1$.

$$\begin{aligned}
&\implies f(a) = t \circ \in (a, a) \\
&\implies \exists x \in A, \forall a \in A,\ f(a) = \in (x, a) \\
&\implies \in (x, a) = t \circ \in (a, a) \\
&\implies \in (a, a) = t \circ \in (a, a)
\end{aligned}$$

i.e. $\in (a, a)$ is a fixed point of t. But this is a contradiction.

This is Russells paradox because by assuming that $\in$ is a weak point surjection we are able to show that

$$a \in a \iff a \notin a$$

where the right hand side is actually $t(a \in a)$.